Phys 410
Fall 2015
Lecture #19 Summary
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The advantages of the Hamiltonian formulation of mechanics: 1) It is the last step before making the observables into operators and developing quantum mechanics, 2) It allows for a powerful geometrical interpretation of classical mechanics in phase space, 3) and this in turn makes it very useful for nonlinear dynamics and statistical mechanics, 4) it is well-suited for applying perturbation theory, 5) it allows for ignorable coordinates to be fully exploited, and 6) the Hamilton equations are first-order-in-time differential equations, which are often easier to solve than the second-order Euler-Lagrange equations.

We considered the Hamiltonian description of a particle moving in one dimension under the influence of a conservative force and showed that Hamilton's equations can be used to reproduce Newton's second law of motion. The procedure of utilizing the Hamiltonian method is: (1) choose the generalized coordinates q_i, (2) write down $T, U,$ and $\mathcal{L}$ in terms of the coordinates and their time-derivatives, (3) compute the conjugate momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, (4) express the $\dot{q}_i$ in terms of q_i *and* p_i, (5) compute the Hamiltonian $\mathcal{H}$, and (6) write out and solve Hamilton's equations. We then used the Hamilton method to solve for the equations of motion of a particle living in two dimensions described by polar coordinates.

The generalized coordinates and their corresponding velocities together represent the state of the system as a single mathematical point in a 2n-dimensional **state space**. The time evolution of the system is represented as a trajectory or orbit in this state space. Unfortunately the Lagrangian does not give a clear set of instructions for how the state point moves in this space, short of solving the equations of motion and calculating the trajectory. (The situation is very different for the Hamiltonian case.) The generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, constitute a set of $2n$ quantities that span **phase space**. The instantaneous state of the entire system is summarized as a single mathematical point in this phase space. Call this point $\vec{z} = (\vec{q}, \vec{p})$, where $\vec{q} = (q_1, \dots q_n)$ is an ordered list of the n generalized coordinates, and $\vec{p} = (p_1, \dots p_n)$ is the list of n conjugate momenta. Hamiltonian's equations describe how this point moves in phase space – in other words it describes how to evolve forward in time by using only "local information", namely your instantaneous location in phase space. Hence the Hamiltonian can be used to find the trajectory of the phase point without first solving the equations of motion. Hamilton's equations are a set of deterministic equations for the evolution of the phase point. It shows that two trajectories that arise from two different initial conditions can never cross, because otherwise there would be two different trajectories arising from the same equation with the same instantaneous value of $\vec{z}$, contrary to the deterministic nature of the phase point evolution equation.

As an example, we considered the $2n = 2-$dimensional phase space of a $n = 1$ one-dimensional harmonic oscillator. The trajectory of the phase point is an ellipse in the (x, p) phase plane.

The Hamiltonian dynamics formulation is useful for quantum mechanics and for statistical mechanics. In quantum mechanics there is an uncertainty principle for a given coordinate and its conjugate momentum, namely $\Delta x \Delta p \geq \hbar/2$. This means that phase space is "chunky" or "fuzzy" on the scale of $\hbar$. This in turn allows for a proper definition of entropy in terms of how many phase space "balls" of dimension $\hbar$ are available to the system.

What follows is a way to "derive" quantum mechanics starting from classical mechanics, following the logic of P. A. M. Dirac. We consider first the Poisson Bracket (PB), which is defined as follows. Consider two dynamical functions of the generalized coordinates and conjugate momenta: $g(\vec{q}, \vec{p})$ and $h(\vec{q}, \vec{p})$. Examples include components of angular momentum, a component of linear momentum, mechanical energy, linear kinetic energy, rotational kinetic energy, etc. Define the PB of g, h as $[g, h] \equiv \sum_{i=1}^{n} \left\{ \frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right\}$. One can show quite easily that the following statements are true about the PB: $\frac{dg}{dt} = [g, \mathcal{H}] + \frac{\partial g}{\partial t}$, $\dot{q}_j = [q_j, \mathcal{H}]$, $\dot{p}_j = [p_j, \mathcal{H}]$, $[q_j, q_k] = 0$, $[p_j, p_k] = 0$, and most interestingly $[q_j, p_k] = \delta_{kj}$. If the PB of two dynamical quantities vanishes, then the quantities are said to **commute**. If the PB of two dynamical quantities is equal to 1, then the quantities are said to be **canonically conjugate**. Any dynamical quantity that commutes with the Hamiltonian and is not explicitly time dependent is a constant of the motion of the system. Knowing about such quantities can be very useful for understanding the motion of a complex system. Also note from the definition of the PB that $[g, h] = -[h, g]$. Starting with this, Dirac noted that the essential new ingredient of quantum mechanics (QM) is that certain observables $(\hat{u}, \hat{v})$ give different answers depending on the order in which the observables operate on a QM system, or in other words $\hat{u}\hat{v} \neq \hat{v}\hat{u}$. To account for this, Dirac re-defined the PB for the quantum case as follows: $i\hbar[u, v]_{QM} \equiv uv - vu$. This leads to the following statements of the "fundamental quantum conditions" for the quantum position and momentum operators: $q_r q_s - q_s q_r = 0$, $p_r p_s - p_s p_r = 0$, and $q_r p_s - p_s q_r = i\hbar\delta_{rs}$. From this statement, one can derive many important results in quantum mechanics, as outlined in Dirac's book *Principles of Quantum Mechanics*.

Why would you want to use the PB in classical mechanics? One reason is that you can calculate the time evolution of any dynamical variable (like g or h discussed above) using the result $\frac{dg}{dt} = [g, \mathcal{H}] + \frac{\partial g}{\partial t}$. The PB can also be used to identify constants of the motion, as discussed in the paragraph above.